

RUBBLING AND OPTIMAL RUBBLING OF GRAPHS

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ABSTRACT. A pebbling move on a graph removes two pebbles at a vertex and adds one pebble at an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed. In this new move one pebble is removed at vertices v and w adjacent to a vertex u and an extra pebble is added at vertex u . A vertex is reachable from a pebble distribution if it is possible to move a pebble to that vertex using rubbling moves. The rubbling number of a graph is the smallest number m needed to guarantee that any vertex is reachable from any pebble distribution of m pebbles. The optimal rubbling number is the smallest number m needed to guarantee a pebble distribution of m pebbles from which any vertex is reachable. We determine the rubbling and optimal rubbling number of some families of graphs including cycles.

1. INTRODUCTION

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. How many pebbles will guarantee that every vertex is reachable, or that all vertices are reachable at the same time? How can we place the smallest number of pebbles such that every vertex is reachable? For a comprehensive list of references for the extensive literature see the survey papers [5, 6].

In the current paper we propose the study of an extension of pebbling called *rubbling*. In this version we also allow a move that removes a pebble from the vertices v and w that are adjacent to a vertex u , and adds a pebble at vertex u . We find rubbling versions of some of the well known pebbling tools such as the transition digraph, the No Cycle Lemma, squishing and smoothing. We use these tools to find the rubbling number and the optimal rubbling number for some families of graphs including complete graphs, complete bipartite graphs, paths, wheels and cycles.

2. PRELIMINARIES

Let G be a simple graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \rightarrow \mathbf{Z}$ where $p(v)$ is the number of pebbles placed at v . A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles

Date: 2/1/2008.

1991 Mathematics Subject Classification. 05C99.

Key words and phrases. pebbling, optimal pebbling, rubbling.

$\sum_{v \in V(G)} p(v)$. We are going to use the notation $p(v_1, \dots, v_n, *) = (a_1, \dots, a_n, q(*))$ to indicate that $p(v_i) = a_i$ for $i \in \{1, \dots, n\}$ and $p(w) = q(w)$ for all $w \in V(G) \setminus \{v_1, \dots, v_n\}$.

Definition 2.1. Consider a pebble function p on the graph G . If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \rightarrow u)$ removes two pebbles at vertex v and adds one pebble at vertex u to create a new pebble function

$$p_{(v, v \rightarrow u)}(v, u, *) = (p(v) - 2, p(u) + 1, p(*)).$$

If $\{w, u\} \in E(G)$ and $v \neq w$ then the *strict rubbing move* $(v, w \rightarrow u)$ removes one pebble each at vertices v and w and adds one pebble at vertex u to create a new pebble function

$$p_{(v, w \rightarrow u)}(v, w, u, *) = (p(v) - 1, p(w) - 1, p(u) + 1, p(*)).$$

A *rubbling move* is either a pebbling move or a strict rubbing move.

Note that the rubbing moves $(v, w \rightarrow u)$ and $(w, v \rightarrow u)$ are the same. Also note that the resulting pebble function might not be a pebble distribution even if p is.

Definition 2.2. A *rubbling sequence* is a finite sequence $s = (s_1, \dots, s_k)$ of rubbing moves. The pebble function gotten from the pebble function p after applying the moves in s is denoted by p_s .

The concatenation of the rubbing sequences $r = (r_1, \dots, r_k)$ and $s = (s_1, \dots, s_l)$ is denoted by $rs = (r_1, \dots, r_k, s_1, \dots, s_l)$.

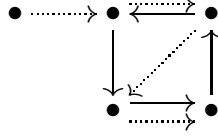
Definition 2.3. A rubbing sequence s is *executable* from the pebble distribution p if $p_{(s_1, \dots, s_i)}$ is nonnegative for all i . A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbing sequence s such that $p_s(v) \geq 1$. The *rubbling number* $\rho(G)$ of a graph G is the minimum number m such that every vertex of G is reachable from any pebble distribution of size m .

A vertex is reachable if a pebble can be moved to that vertex using rubbing moves with actual pebbles without ever running out of pebbles. Changing the order of moves in an executable rubbing sequence s may result in a sequence r that is no longer executable. On the other hand the ordering of the moves has no effect on the resulting pebble function, that is, $p_s = p_r$. This justifies the following definition.

Definition 2.4. Let S be a multiset of rubbing moves. The pebble function gotten from the pebble function p after applying the moves in S in any order is denoted by p_S .

3. THE TRANSITION DIGRAPH AND THE NO CYCLE LEMMA

Definition 3.1. Given a multiset S of rubbing moves on G , the *transition digraph* $T(G, S)$ is a directed multigraph whose vertex set is $V(G)$, and each move $(v, w \rightarrow u)$ in S is represented by two directed edges (v, u) and (w, u) . The transition digraph of a rubbing sequence $s = (s_1, \dots, s_n)$ is $T(G, s) = T(G, S)$, where $S = \{s_1, \dots, s_n\}$ is the multiset of moves in s . Let $d_{T(G, S)}^-$ represent the in-degree and $d_{T(G, S)}^+$ the out-degree in $T(G, S)$. We simply write d^- and d^+ if the transition digraph is clear from context.

FIGURE 3.1. Arrows of $T(G, Q)$. The solid arrows belong to C .

The transition digraph only depends on the rubbling moves and the graph but not on the pebble distribution or on the order of the moves. It is possible that $T(G, S) = T(G, R)$ even if $S \neq R$. If $T(G, S) = T(G, R)$ then $p_S = p_R$, so the effect of a rubbling sequence on a pebble function only depends on the transition digraph. In fact we have the following.

Lemma 3.2. *If p is a pebble function on G and S is a multiset of rubbling moves then*

$$p_S(v) = p(v) + d^-(v)/2 - d^+(v)$$

for all $v \in V(G)$.

Proof. The three terms on the right hand side represent the original number of pebbles, the number of pebbles arrived at v and the number of pebbles moved away from v . \square

We are often interested in the value of $q_R(v) - p_S(v)$. The function Δ defined in the following lemma is going to simplify our notation. The three parameters of Δ represent the change in the number of pebbles, the change in the in-degree and the change in the out-degree. The proof is a trivial calculation.

Lemma 3.3. *Define $\Delta(a, b, c) = a + b/2 - c$. Then*

$$q_R(v) - p_S(v) = \Delta(q(v) - p(v), d_{T(G,R)}^-(v) - d_{T(G,S)}^-(v), d_{T(G,R)}^+(v) - d_{T(G,S)}^+(v)).$$

If the rubbling sequence s is executable from a pebble distribution p then we must have $p_s \geq 0$. This motivates the following terminology.

Definition 3.4. A multiset S of rubbling moves on G is *balanced* with a pebble distribution p at vertex v if $p_S(v) \geq 0$. We say S is *balanced* with p if S is balanced with p at all $v \in V(G)$, that is, $p_S \geq 0$. We say that a rubbling sequence s is balanced with p if the multiset of moves in s is balanced with p .

S is trivially balanced with a pebble distribution at v if $d_{T(G,S)}^+(v) = 0$. The balance condition is necessary but not sufficient for a rubbling sequence to be executable. The pebble distribution $p(u, v, w) = (1, 1, 1)$ on the cycle C_3 is balanced with $s = ((u, u \rightarrow v), (v, v \rightarrow w), (w, w \rightarrow u))$, but s is not executable. The problem is caused by the cycle in the transition digraph. The goal of this section is to overcome this difficulty.

Definition 3.5. A multiset of rubbling moves or a rubbling sequence is called *acyclic* if the corresponding transition digraph has no directed cycles. Let S be a multiset of rubbling moves. An acyclic multiset $R \subseteq S$ is called an *untangling* of S if $p_R \geq p_S$.

Proposition 3.6. *Every multiset of rubbling moves has an untangling.*

Proof. Let S be the multiset of rubbing moves. Suppose that $T(G, S)$ has a directed cycle C . Let Q be the multiset of elements of S corresponding to the arrows of C , see Figure 3.1. We show that $p_R \geq p_S$ where $R = S \setminus Q$. If $v \in V(C)$ then there is an $a \leq -1$ such that

$$p_R(v) - p_S(v) = \Delta(0, -2, a) = -1 - a \geq 0.$$

If $v \in V(G) \setminus V(C)$ then there is an $a \leq 0$ such that

$$p_R(v) - p_S(v) = \Delta(0, 0, a) \geq 0.$$

We can repeat this process on R until we eliminate all the cycles. This can be finished in finitely many steps since every step decreases the number of edges in R . The resulting multiset is an untangling of S . \square

Note that a multiset of moves can have several untanglings. Also note that if a pebble distribution p is balanced with S and R is an untangling of S then $p_R \geq p_S \geq 0$ and so p is also balanced with R .

Lemma 3.7. *If p is a pebble distribution on G that is balanced with the multiset S of moves and $t = (v, w \rightarrow u) \in S$ such that $d^-(v) = 0 = d^-(w)$ then t is executable from p .*

Proof. If $v \neq w$ then $p(v) \geq d^+(v) \geq 1$ and $p(w) \geq d^+(w) \geq 1$. If $v = w$ then $p(v) \geq d^+(v) \geq 2$. In both cases s is executable from p . \square

Proposition 3.8. *If the pebble distribution p on G is balanced with the acyclic multiset S of rubbing moves then there is a sequence s of the elements of S such that s is executable from p .*

Proof. We define s recursively. Let $R_1 = S$. Since R_1 is acyclic, we must have a move $s_1 = (v_1, w_1 \rightarrow u_1) \in R_1$ such that $d_{T(G, R_1)}^-(v_1) = 0 = d_{T(G, R_1)}^-(w_1)$. Then s_1 is executable from p by Lemma 3.7. Let $R_i = R_{i-1} \setminus \{s_{i-1}\}$. Then R_i is acyclic so we must have a move $s_i = (v_i, w_i \rightarrow u_i) \in R_i$ such that $d_{T(G, R_i)}^-(v_i) = 0 = d_{T(G, R_i)}^-(w_i)$. Then $p_{(s_1, \dots, s_{i-1})}$ is balanced with R_i since $(p_{(s_1, \dots, s_{i-1})})_{R_i} = p_S \geq 0$ and so s_i is executable from $p_{(s_1, \dots, s_{i-1})}$. The sequence $s = (s_1, \dots, s_{|S|})$ is an ordering of the elements of S that is executable from p . \square

The following is the rubbing version of the No-Cycle Lemma for pebbling [3, 7, 8].

Lemma 3.9. (No Cycle) *Let p be a pebble distribution on G and $v \in V(G)$. The following are equivalent.*

- (1) v is reachable from p .
- (2) There is a multiset S of rubbing moves such that S is balanced with p and $p_S(v) \geq 1$.
- (3) There is an acyclic multiset R of rubbing moves such that R is balanced with p and $p_R(v) \geq 1$.
- (4) v is reachable from p through an acyclic rubbing sequence.

Proof. If v is reachable from p then there is an executable sequence s of rubbing moves. The multiset S of rubbing moves of s is balanced with p and $p_S(v) \geq 1$. So (1) implies (2). If S satisfies (2) then an untangling R of S satisfies (3). Suppose R satisfies (3). By Proposition 3.8, there is an executable ordering r of the moves of R .

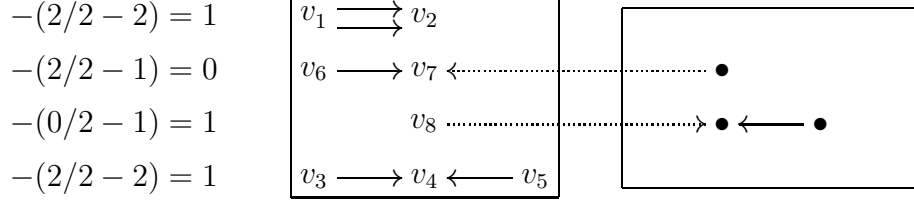


FIGURE 4.1. Arrows in $T(G, S)$ representing the possible types of rubbing moves in E . The vertices in the same box are equivalent. The solid arrows connect equivalent vertices. The calculation on the left shows the change in $\sum_i (\frac{1}{2}d^-(v_i) - d^+(v_i))$ after the removal of one of the rubbing moves.

This r is acyclic and v is reachable through r since $p_r(v) = p_R(v) \geq 1$. So (3) implies (4). Finally, (4) clearly implies (1). \square

Corollary 3.10. *If a vertex is reachable from a pebble distribution p on G then it is also reachable by a rubbing sequence in which no move of the form $(v, a \rightarrow u)$ is followed by a move of the form $(u, b \rightarrow v)$.*

4. BASIC RESULTS

It is clear from the definition that for all graphs G we have $\rho(G) \leq \pi(G)$ where π is the pebbling number. For the pebbling number we have $2^{\text{diam}(G)} \leq \pi(G)$. This is also true for the rubbing number. To see this we need to find the rubbing number of a path first.

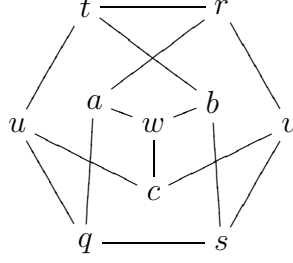
Proposition 4.1. *The rubbing number of the path with n vertices is $\rho(P_n) = 2^{n-1}$.*

Proof. Let v_1, \dots, v_n be the consecutive vertices of P_n . Let $p(v_n, *) = (m, 0)$ be a pebble distribution from which v_1 is reachable through the acyclic rubbing sequence s . We show that $m \geq 2^{n-1}$. Since v_1 is reachable and $p(v_1) = 0$, the balance condition at v_1 implies that $T(G, s)$ has at least 2 arrows from v_2 to v_1 and so $d^+(v_2) \geq 2$. Since $T(G, s)$ has no cycles, there are no arrows from v_1 to v_2 . The balance condition at v_2 now implies that $T(G, s)$ has at least 4 arrows from v_3 to v_2 and so $d^+(v_3) \geq 2^2$. An inductive argument shows that $d^+(v_n) \geq 2^{n-1}$ and $d^-(v_n) = 0$. The balance condition at v_n implies that $m \geq d^+(v_n) \geq 2^{n-1}$. This shows that $2^{n-1} \leq \rho(P_n)$.

It is known [5] that $\pi(P_n) = 2^{n-1}$. The result now follows from the inequality $2^{n-1} \leq \rho(P_n) \leq \pi(P_n) = 2^{n-1}$. \square

Proposition 4.2. *If the graph G has diameter d then $2^d \leq \rho(G)$.*

Proof. Let v_0 and v_d be vertices at distance d . Let $p(v_0, *) = (m, 0)$ be a pebble distribution from which v_d is reachable through the rubbing sequence s . We now build a quotient rubbing problem. Let $[v]$ be the equivalence class of v in the partition of the vertices of G according to their distances from v_0 . The quotient simple graph H is isomorphic to P_{d+1} with leafs $[v_0] = \{v_0\}$ and $[v_d]$. Let $q([v]) = \sum_{w \in [v]} p(w)$ for all $[v] \in V(H)$ and note that $q([v_0], *) = (m, 0)$. The rubbing sequence s induces a multiset R of rubbing moves on H . We construct this R from the multiset S of rubbing moves of s . Let E be the multiset of moves of S of the form $(v, w \rightarrow u)$ where

FIGURE 4.2. The Petersen graph P .

$v \in [u]$ or $w \in [u]$. Define R to be the multiset of moves of the form $([v], [w] \rightarrow [u])$ where $(v, w \rightarrow u)$ runs through the elements of $S \setminus E$.

We show that R is balanced with q . Figure 4.1 shows the possible types of moves in E . The removal of any of these moves does not decrease the value of $\sum_{v_i \in [v]} (\frac{1}{2}d^-(v_i) - d^+(v_i))$ and so

$$q_R([v]) = \sum_{v_i \in [v]} p_{S \setminus E}(v_i) \geq \sum_{v_i \in [v]} p_S(v_i) \geq 0$$

since p is balanced with S .

We also have $q_R([v_d]) \geq 1$ since v_d is reachable and so $p_S(v_d) \geq 1$. Thus $[v_d]$ is reachable from q and so the result now follows from Proposition 4.1. \square

For the pebbling number we have $\pi(G) \geq |V(G)|$. This inequality does not hold for the rubbing number as we can see in the next result.

Proposition 4.3. *We have the following values for the rubbing number:*

- a. $\rho(K_n) = 2$ for $n \geq 2$ where K_n is the complete graph with n vertices;
- b. $\rho(W_n) = 4$ for $n \geq 4$ where W_n is the wheel with n spikes;
- c. $\rho(K_{m,n}) = 4$ for $m, n \geq 2$ where $K_{m,n}$ is a complete bipartite graph;
- d. $\rho(Q^n) = 2^n$ for $n \geq 1$ where Q^n is the n -dimensional hypercube;
- e. $\rho(G) = 2^{s+1}$ where s is the number of vertices in the spine of the caterpillar G .

Proof. a. A single pebble is clearly not sufficient but any vertex is reachable with two pebbles using a single move.

b. If we have 4 pebbles then we can move 2 pebbles to the center using two moves. Then any other vertex is reachable from the center in a single move. On the other hand $\rho(W_n) \geq 2^{\text{diam}(W_n)} = 2^2 = 4$.

c. It is easy to see that from any pebble distribution of size 4 any vertex is reachable in at most 3 moves. On the other hand we have $\rho(K_{m,n}) \geq 2^{\text{diam}(K_{m,n})} = 2^2 = 4$.

d. We know [2] that $\pi(Q^n) = 2^n$. The result now follows from the inequality $2^n = 2^{\text{diam}(Q^n)} \leq \rho(Q^n) \leq \pi(Q^n) = 2^n$.

e. The result follows easily from Proposition 4.1. \square

Proposition 4.4. *The rubbing number of the Petersen graph P is $\rho(P) = 5$.*

Proof. Consider Figure 4.2. It is easy to see that vertex w is not reachable from the pebble distribution $p(r, s, *) = (3, 1, 0)$ and so $\rho(P) > 4$. To show that $\rho(P) \leq 5$,

assume that a vertex is not reachable from a pebble distribution p of size 5. Since P is vertex transitive, we can assume that this vertex is w . Then we must have

$$p(a) + p(b) + p(c) + \left\lfloor \frac{p(q) + p(r)}{2} \right\rfloor + \left\lfloor \frac{p(s) + p(t)}{2} \right\rfloor + \left\lfloor \frac{p(u) + p(v)}{2} \right\rfloor \leq 1,$$

otherwise we could make the total number of pebbles at vertices a, b and c more than 2 after which w is reachable. This inequality forces $p(a) = p(b) = p(c) = 0$ and two of the remaining terms to be 0 as well. So by symmetry we can assume that the last term is 1 and all the other terms are 0. Then we must have $p(u) + p(v) = 3$ and $p(q) + p(r) = 1 = p(s) + p(t)$. A simple case analysis shows that w is reachable from this p , which is a contradiction. \square

5. SQUISHING

The following terms are needed for the rubbling version of the squishing lemma of [1]. A *thread* in a graph is a path containing vertices of degree 2. A pebble distribution is *squished* on a thread P if all the pebbles on P are placed on a single vertex of P or on two adjacent vertices of P .

Lemma 5.1. *Let P be a thread in G . If vertex $x \notin V(P)$ is reachable from the pebble distribution p then x is reachable from p through a rubbling sequence in which there is no strict rubbling move of the form $(v, w \rightarrow u)$ where $u \in V(P)$.*

Proof. Let S be an acyclic multiset of rubbling moves balanced with p such that $p_S(x) \geq 1$. Let E be the multiset of strict rubbling moves of S of the form $(v, w \rightarrow u)$ where $u \in V(P)$.

If $e = (v, w \rightarrow u) \in E$ then we have $d_{T(G, S \setminus \{e\})}^+(u) = d_{T(G, S)}^+(u) = 0$ since S is acyclic and so $S \setminus \{e\}$ is balanced with p at u . It is clear that $p_{S \setminus \{e\}}(y) \geq p_S(y)$ for all $y \in V(G) \setminus \{u\}$ and so $S \setminus \{e\}$ is balanced with p . We still know that $S \setminus \{e\}$ is acyclic and $p_{S \setminus \{e\}}(x) \geq 1$, so induction shows that $R = S \setminus E$ is balanced with p .

By Proposition 3.8, there is an ordering r of the elements of R that is executable from p . Then v is reachable through r since $p_r(v) = p_S(v) \geq 1$. \square

The following is the rubbling version of the Squishing Lemma for pebbling [1].

Lemma 5.2. (Squishing) *If vertex v is not reachable from a pebble distribution with size n then there is a pebble distribution r of size n that is squished on each thread not containing v such that v is not reachable from r either.*

Proof. The result follows from [1, Lemma 4] and 5.1. \square

6. RUBBLING C_n

The Squishing Lemma allows us to find the rubbling numbers of cycles. For the pebbling numbers of C_n see [10, 1].

Proposition 6.1. *The rubbling number of an even cycle is $\rho(C_{2k}) = 2^k$.*

Proof. It is well known [10] that $\pi(C_{2k}) = 2^k$. The first result now follows since

$$2^k = 2^{\text{diam}(C_{2k})} \leq \rho(C_{2k}) \leq \pi(C_{2k}) = 2^k.$$

\square

Proposition 6.2. *The rubbing number of an odd cycle is $\rho(C_{2k+1}) = \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor + 1$.*

Proof. Let C_{2k+1} be the cycle with consecutive vertices

$$x_k, x_{k-1}, \dots, x_1, v, y_1, y_2, \dots, y_k, x_k.$$

First we show that $\rho(C_{2k+1}) \leq \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor + 1$. Let p be a pebble distribution on C_{2k+1} from which not every vertex is reachable. It suffices to show that p contains at most $\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$ pebbles. By symmetry, we can assume that v is the vertex that is not reachable from p . By the Squishing Lemma, we can assume that p is squished on the thread with consecutive vertices $y_1, \dots, y_k, x_k, \dots, x_1$.

First we consider the case when all the pebbles are at distance k from v , that is, $p(x_k, y_k, *) = (a, b, 0)$. By symmetry, we can assume that $0 \leq a \leq b$. Then we must have

$$(6.1) \quad \left\lfloor \frac{a}{2} \right\rfloor + b \leq 2^k - 1,$$

otherwise we could move $\lfloor \frac{a}{2} \rfloor$ pebbles from vertex x_k to vertex y_k and then reach v from b_k . Hence $\frac{a}{2} < \lfloor \frac{a}{2} \rfloor + 1 \leq 2^k - 1 - b + 1 = 2^k - b$ and so

$$(6.2) \quad a + 2b \leq 2^{k+1} - 1.$$

We also must have

$$(6.3) \quad \left\lfloor \frac{b - 2^{k-1}}{2} \right\rfloor + a \leq 2^{k-1} - 1,$$

otherwise we could move $\lfloor \frac{b - 2^{k-1}}{2} \rfloor$ pebbles from vertex y_k to vertex x_k after which x_1 is reachable from x_k and y_1 is reachable from y_k , and so v would be reachable by the move $(x_1, y_1 \rightarrow v)$. Hence $\frac{b - 2^{k-1}}{2} < \lfloor \frac{b - 2^{k-1}}{2} \rfloor + 1 \leq 2^{k-1} - 1 - a + 1 = 2^{k-1} - a$ and so

$$(6.4) \quad b + 2a \leq 2^k + 2^{k-1} - 1.$$

Adding (6.2) and (6.4) gives

$$3(a + b) \leq 2^{k+1} - 1 + 2^k + 2^{k-1} - 1 = 7 \cdot 2^{k-1} - 2,$$

which shows that $|p| = a + b \leq \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$.

Now we consider the case when some pebbles are closer to v than k , that is, $p(x_i, x_{i+1}, *) = (b, a, 0)$ with $b \geq 1$ and $a \geq 0$ for some $1 \leq i < k$. Then we must have $\lfloor \frac{a}{2} \rfloor + b \leq 2^i - 1 \leq 2^{k-1} - 1$ otherwise v is reachable. Hence

$$\begin{aligned} |p| &= a + b \leq a - \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{a}{2} \right\rfloor + b \\ &\leq \left\lfloor \frac{a}{2} \right\rfloor + 1 + 2^{k-1} - 1 \leq 2^{k-1} - 1 - b + 1 + 2^{k-1} - 1 \\ &= 2 \cdot 2^{k-1} - 2 < \left\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \right\rfloor. \end{aligned}$$

Now we show that we can always distribute $\lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor$ pebbles so that v is unreachable and so $\rho(C_{2k+1}) \geq \lfloor \frac{7 \cdot 2^{k-1} - 2}{3} \rfloor + 1$. Let $a = \lfloor \frac{2^k}{3} \rfloor$ and $b = \lfloor \frac{5 \cdot 2^{k-1}}{3} \rfloor$. It is easy to

check that

$$a = \begin{cases} \frac{2^k-2}{3}, & k \text{ odd} \\ \frac{2^k-1}{3}, & k \text{ even} \end{cases}, \quad b = \begin{cases} \frac{5 \cdot 2^{k-1}-2}{3}, & k \text{ odd} \\ \frac{5 \cdot 2^{k-1}-1}{3}, & k \text{ even} \end{cases}, \quad \left\lfloor \frac{7 \cdot 2^{k-1}-2}{3} \right\rfloor = \begin{cases} \frac{7 \cdot 2^{k-1}-4}{3}, & k \text{ odd} \\ \frac{7 \cdot 2^{k-1}-2}{3}, & k \text{ even} \end{cases}$$

and so $a+b = \lfloor \frac{7 \cdot 2^{k-1}-2}{3} \rfloor$. We show that v is unreachable from the pebble distribution $p(x_k, y_k, *) = (a, b, 0)$.

It is easy to see that a and b satisfy (6.2) and (6.4). Suppose that v is reachable from p , that is, there is an acyclic multiset S of rubbing moves that is balanced with p satisfying $p_S(v) \geq 1$. The balance condition at v shows that $d^-(v) \geq 2$. Hence S must have at least one of $(x_1, y_1 \rightarrow v)$, $(x_1, x_1 \rightarrow v)$ or $(y_1, y_2 \rightarrow v)$.

First assume that $(x_1, y_1 \rightarrow v) \in S$. The argument used in the proof of Proposition 4.1 shows that then $T(G, S)$ has at least 2^{i-1} arrows from x_i to x_{i-1} and from y_i to y_{i-1} for all $i \in \{2, \dots, k\}$. Since S is acyclic, any arrow in $T(G, S)$ pointing to x_k must come from y_k . So the balance condition at x_k requires m arrows from y_k to x_k satisfying $2^{k-1} \leq a + \frac{m}{2}$. The balance condition at y_k gives $2^{k-1} + m \leq b$. Combining the two inequalities gives $2^k + 2^{k-1} \leq b + 2a$ which contradicts (6.4).

Next assume that $(y_1, y_1 \rightarrow v) \in S$. Then $T(G, S)$ has at least 2^i arrows from y_i to y_{i-1} for all $i \in \{2, \dots, k\}$. The balance condition at y_k requires m arrows from x_k to y_k satisfying $2^k \leq b + \frac{m}{2}$. We must have $d^-(x_k) = 0$, otherwise there is a directed path from v to x_k which is impossible since S is acyclic. The balance condition at x_k gives $m \leq a$. Combining the two inequalities gives $2^{k+1} \leq a + 2b$ which contradicts (6.2).

Similar argument shows that $(x_1, x_1 \rightarrow v) \in S$ is also impossible. \square

7. OPTIMAL RUBBLING

Optimal pebbling was studied in [10, 9, 4, 1]. In this section we investigate the optimal rubbing number of certain graphs.

Definition 7.1. The *optimal rubbing number* $\rho_{\text{opt}}(G)$ of a graph G is the minimum number m for which there is a pebble distribution of size m from which every vertex of G is reachable.

Proposition 7.2. We have the following values for the optimal rubbing number:

- a. $\rho_{\text{opt}}(K_n) = 2$ for $n \geq 2$ where K_n is the complete graph with n vertices;
- b. $\rho_{\text{opt}}(W_n) = 2$ for $n \geq 4$ where W_n is the wheel with n spikes;
- c. $\rho_{\text{opt}}(K_{m,n}) = 3$ for $m, n \geq 3$ where $K_{m,n}$ is the complete bipartite graph;
- d. $\rho_{\text{opt}}(P) = 4$ where P is the Petersen graph.

Proof. a. Not every vertex of K_n is reachable from a distribution of size 1 since $n \geq 2$. On the other hand any vertex is reachable by a single move from any distribution of size 2.

b. Again, not every vertex of W_n is reachable from a distribution of size 1. On the other hand, every vertex is reachable from the distribution that has 2 pebbles at the center of W_n .

c. Let A and B be the natural partition of the vertex set of $K_{m,n}$. Let p be a pebble distribution of size 2. If p places both pebbles on vertices in A then there is a vertex in A that is not reachable from p . If p places both pebbles on vertices in B

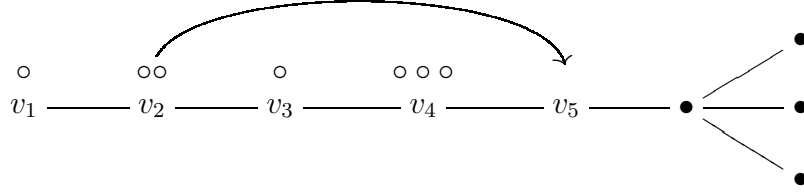


FIGURE 7.1. Visualization of a single rolling move with $i = 2$ and $n = 5$. An arrow indicates the transfer of a single pebble

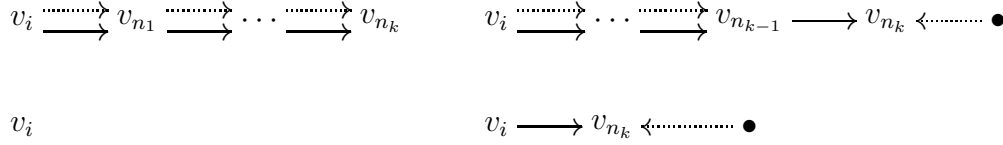


FIGURE 7.2. Four possible configurations for $T(G, S \setminus R)$. The solid arrows represent the arrows of P .

then there is a vertex in B that is not reachable from p . If p places one pebble on a vertex in A and one pebble on a vertex in B then both A and B have vertices that are unreachable from p . On the other hand any vertex is reachable in at most two moves from a pebble distribution that places one pebble on a vertex in A and two pebbles on a vertex in B .

d. Every vertex is reachable from the pebble distribution that has 4 pebbles on any of the vertices. A simple case analysis shows that 3 pebbles are not sufficient to make every vertex reachable. \square

Rolling moves serve the same purpose as the smoothing move of [1].

Definition 7.3. Let v_1, \dots, v_n be the consecutive vertices of a path such that the degree of v_1 is 1 and the degrees of v_2, v_3, \dots, v_{n-1} are all 2. The subgraph induced by $\{v_1, \dots, v_n\}$ is called an *arm* of the graph. Let p be a pebble distribution such that $p(v_i) \geq 2$ for some $i \in \{1, \dots, n-1\}$, $p(v_n) = 0$, and $p(v_j) \geq 1$ for all $j \in \{1, \dots, n-1\}$. A *single rolling move* creates a new pebble distribution q by taking one pebble from v_i and placing it on v_n , that is $q(v_i, v_n, *) = (p(v_i) - 1, 1, p(*))$. See Figure 7.1.

Lemma 7.4. Let q be a pebble distribution on G gotten from the pebble distribution p by applying a single rolling move from v_i to v_n on the arm with vertices v_1, \dots, v_n . If vertex $u \in G$ is reachable from p then u is also reachable from q .

Proof. If u is a vertex of the arm then it is clearly reachable from q so we can assume that u is not on the arm. Let S be an acyclic multiset of rubbing moves balanced with p such that $p_S(u) \geq 1$. Let P be a maximum length directed path in $T(G, S)$ starting at v_i and not going further than v_n . Then P has consecutive vertices $v_i = v_{n_0}, v_{n_1}, \dots, v_{n_k}$ on the arm. Let R be the multiset containing the elements of S without the moves corresponding to the arrows of P . We show that R is balanced with q and so u is reachable from q since $q_R(u) = p_S(u) \geq 1$. Figure 7.2 shows the possible configurations for $T(G, S \setminus R)$. We have $d_{T(G, S)}^+(v_{n_k}) = 0$ even if $n_k = 1$. If $n_k = n$ then

$$q_R(v_{n_k}) = p_S(v_{n_k}) + \Delta(1, -2, 0) = p_S(v_{n_k}) \geq 1 \geq 0,$$

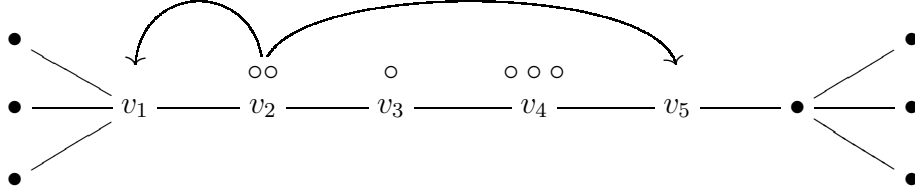


FIGURE 7.3. Visualization of a double rolling move with $i = 2$ and $n = 5$. An arrow indicates the transfer of a single pebble.

while if $n_k \neq n$ then

$$q_R(v_{n_k}) = p_S(v_{n_k}) + \Delta(0, -2, 0) \geq p_S(v_{n_k}) - 1 \geq 2 - 1 \geq 0.$$

So R is balanced with q at v_{n_k} . If $d_{T(G,S)}^+(v_{n_0}) = 0$ then $n_0 = n_k$, otherwise there is an $a \in \{-1, -2\}$ such that

$$q_R(v_{n_0}) = p_S(v_{n_0}) + \Delta(-1, 0, a) \geq p_S(v_{n_0}) \geq 0$$

and so R is balanced with q at v_{n_0} . If $0 < j < k$ then there is an $a \in \{-1, -2\}$ such that

$$q_R(v_{n_j}) = p_S(v_{n_j}) + \Delta(0, -2, a) \geq p_S(v_{n_j}) \geq 0$$

and so R is balanced with q at v_{n_j} . It is clear that R is balanced with q at every other vertex. \square

Definition 7.5. Let v_1, \dots, v_n be the consecutive vertices of a path such that the degrees of v_2, v_3, \dots, v_{n-1} are all 2. Let p be a pebble distribution such that $p(v_1) = 0 = p(v_n)$, $p(v_i) \geq 2$ for some $i \in \{2, \dots, n-1\}$ and $p(v_j) \geq 1$ for all $j \in \{2, \dots, n-1\}$. A *double rolling move* creates a new pebble distribution q by taking two pebbles from v_i and placing one pebble on v_1 and one pebble on v_n , that is $q(v_i, v_1, v_n, *) = (p(v_i) - 2, 1, 1, p(*))$. See Figure 7.3.

Lemma 7.6. Let q be a pebble distribution on G gotten from the pebble distribution p by applying a double rolling move from vertex v_i to vertices v_1 and v_n on the path with consecutive vertices v_1, \dots, v_n . If vertex $u \in G$ is reachable from p then u is also reachable from q .

Proof. If $u \in \{v_1, \dots, v_n\}$ then it is clearly reachable from q so we can assume that $u \notin \{v_1, \dots, v_n\}$. Let S be an acyclic multiset of rubbing moves balanced with p such that $p_S(u) \geq 1$. Let P be a maximum length directed path in $T(G, S)$ starting at v_i and not going further than v_1 or v_n . Then P has consecutive vertices $v_i = v_{n_0}, v_{n_1}, \dots, v_{n_k} \in \{v_1, \dots, v_n\}$. Let R be the multiset containing the elements of S without the moves corresponding to the arrows of P . An argument similar to the one in the proof of Lemma 7.4 shows that R is clearly balanced with q at every vertex except maybe at v_i . If $n_k = n_0$ or the arrow (v_{n_0}, v_{n_1}) in P corresponds to a pebbling move, then R is balanced with q at v_i as well. Then u is reachable from q since $q_R(u) = p_S(u) \geq 1$.

So we can assume that (v_{n_0}, v_{n_1}) corresponds to a strict rubbing move and that $k = 1$. Let \tilde{P} be a maximum length path in $T(G, R)$. Since $k = 1$, the length of \tilde{P} is either 0 or 1. If this length is 0, then q is balanced with R at v_i since $d_{T(G,R)}^+(v_i) = 0$ and we are done. If the length of \tilde{P} is 1, then let \tilde{R} be the multiset containing the

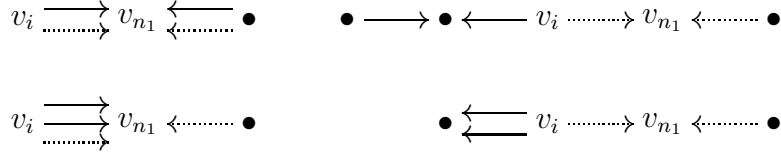


FIGURE 7.4. The four possible configurations for $T(G, S \setminus \tilde{R})$. The solid arrows represent the moves corresponding to the arrows of \tilde{P} . The dotted arrows represent the moves corresponding to the arrows of P .

elements of R without the moves corresponding to the arrows of \tilde{P} . Figure 7.4 shows the possibilities for $T(G, S \setminus \tilde{R})$. It is easy to check that \tilde{R} is balanced with q in each case. Thus u is reachable from q since $q_{\tilde{R}}(u) \geq p_S(u)$. \square

Rolling moves make it possible to find the optimal rubbling number of paths and cycles.

Proposition 7.7. The optimal rubbling number of the path is $\rho_{\text{opt}}(P_n) = \lceil \frac{n+1}{2} \rceil$.

Proof. Let P_n be the path with consecutive vertices v_1, \dots, v_n . It is clear that every vertex is reachable from the pebble distribution

$$p(v_i) = \begin{cases} 1, & i \text{ is odd or } i = n \\ 0, & \text{else} \end{cases}$$

which has size $\lceil \frac{n+1}{2} \rceil$.

Now assume that there is a pebble distribution of size $\lceil \frac{n+1}{2} \rceil - 1$ from which every vertex of P_n is reachable. Let us apply all available rolling moves (single or double). The process ends in finitely many steps since a rolling move reduces the number of pebbles on vertices with more than one pebble by at least one. If there is a vertex with more than one pebble and a vertex with no pebbles, then a rolling move is available. The number of pebbles is not larger than the number of vertices, so the resulting pebble distribution q has at most one pebble on each vertex. Every vertex of P_n still must be reachable from q by Lemma 7.6.

The only moves executable directly from q are strict rubbling moves. By the No Cycle Lemma we can assume that every vertex is reachable by a sequence of moves in which a strict rubbling move $(x, y \rightarrow z)$ is not followed by a move of the form $(z, z \rightarrow x)$ or $(z, z \rightarrow y)$. So we can assume that every vertex is reachable through strict rubbling moves. Then we must have $q(v_1) = 1 = q(v_n)$ otherwise v_1 or v_n is not reachable. A pigeon hole argument shows that there must be two neighbor vertices u and w such that $q(u) = 0 = q(w)$. But then neither u nor w is reachable from q , which is a contradiction. \square

Proposition 7.8. The optimal rubbling number of the cycle is $\rho_{\text{opt}}(C_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 3$.

n	2	3	4	5
$\rho(B_n)$	4	16	> 23	
$\rho_{\text{opt}}(B_n)$	2	4	6	
$\rho_{\text{opt}}(Q^n)$	2	3	4	6

TABLE 1. Rubbling values without a known general formula.

Proof. Let C_n be the cycle with consecutive vertices v_1, \dots, v_n . It is clear that every vertex is reachable from the pebble distribution

$$p(v_i) = \begin{cases} 1, & i \text{ is odd} \\ 0, & \text{else} \end{cases}$$

which has size $\lceil \frac{n}{2} \rceil$.

Now assume that there is a pebble distribution of size $\lceil \frac{n}{2} \rceil - 1$ from which every vertex of C_n is reachable. Let us apply all available double rolling moves. The process ends in finitely many steps since a double rolling move reduces the number of pebbles on vertices with more than one pebble by two. If there is a vertex with more than one pebble and two vertices with no pebbles, then a double rolling move is available. The number of pebbles is smaller than the number of vertices, so the resulting pebble distribution q has at most one pebble on each vertex. Every vertex of C_n still must be reachable from q .

The only moves executable directly from q are strict rubbling moves. The No Cycle Lemma implies that we can assume that every vertex is reachable through strict rubbling moves. A pigeon hole argument shows that there must be two neighbor vertices u and w such that $q(u) = 0 = q(w)$. But then neither u nor w is reachable from q which is a contradiction. \square

8. FURTHER QUESTIONS

There are plenty of unanswered questions. The following might not be too hard to answer.

- What is the optimal rubbling number for the hypercube Q^n . It is fairly easy to get answers for small n with a computer. The known values are listed in Table 1.
- Does Graham's conjecture hold for the rubbling number?
- Is the cover rubbling number the same as the cover pebbling number for every graph?
- We have $\pi(P_n) = \rho(P_n)$, $\pi(Q^n) = \rho(Q^n)$ and it is easy to check that $\pi(L) = 8 = \rho(L)$ where L is the Lemke graph [6]. This is not always the case though. Is it possible to characterize those graphs for which the pebbling and the rubbling numbers are the same?
- Let $f(d, n) = \max\{\rho(G) \mid |V(G)| = n \text{ and } \text{diam}(G) = d\}$. It is not hard to check that $f(2, n) \leq 5$ and $f(3, n) \leq 9$ for $n \in \{1, \dots, 7\}$. Do these upper limits hold for all n ? Is it true that $f(d, n) \leq 2^d + 1$ for all d and n ?

REFERENCES

1. David P. Bunde, Erin W. Chambers, Daniel Cranston, Kevin Milans, and Douglas B. West, *Pebbling and optimal pebbling in graphs*, (Preprint).
2. Fan R. K. Chung, *Pebbling in hypercubes*, SIAM J. Discrete Math. **2** (1989), no. 4, 467–472.
3. Betsy Crull, Tammy Cundiff, Paul Feltman, Glenn H. Hurlbert, Lara Pudwell, Zsuzsanna Szaniszló, and Zolt Tuza, *The cover pebbling number of graphs*, Discrete Math. **296** (2005), no. 1, 15–23.
4. Hung-Lin Fu and Chin-Lin Shiue, *The optimal pebbling number of the complete m -ary tree*, Discrete Math. **222** (2000), no. 1-3, 89–100.
5. Glenn Hurlbert, *A survey of graph pebbling*, Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1999), vol. 139, 1999, pp. 41–64.
6. ———, *Recent progress in graph pebbling*, Graph Theory Notes N. Y. **49** (2005), 25–37.
7. Kevin Milans and Bryan Clark, *The complexity of graph pebbling*, arxiv.org/abs/math/0503698.
8. David Moews, *Pebbling graphs*, J. Combin. Theory Ser. B **55** (1992), no. 2, 244–252.
9. ———, *Optimally pebbling hypercubes and powers*, Discrete Math. **190** (1998), no. 1-3, 271–276.
10. Lior Pachter, Hunter S. Snevily, and Bill Voxman, *On pebbling graphs*, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), vol. 107, 1995, pp. 65–80.

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